Age-dependent diffusive Lotka-Volterra type systems

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Abstract

In this paper it is shown that the sub-supersolution method works for age-dependent diffusive nonlinear systems with non-local initial conditions. As application, we prove the existence and uniqueness of positive solution for a kind of Lotka-Volterra systems, as well as the blow-up in finite time in a particular case.

Key Words. Sub and supersolutions, Lotka-Volterra systems, backward problem, blow-up.

AMS Classification. 35B05, 35K57, 35L60, 92D25

1 Introduction

The introduction of the age-structure in the population dynamic models supposes a considerable advance insofar as it permits the dependence of the age of parameters so sensitive to it as the birth and mortality rates are. From the mathematical point of view, the combination of the equation and, mainly, the nonlocal initial condition for the age presents many interesting and nontrivial questions.

After, this structure has been exploited to describe the evolution of a population divided in two sub-populations in, for instance, the frame of the epidemics theory; in this case, the age structure is rather a contagion-time structure.

[1]Supported by the Spanish Ministry of Science and Technology under Grant BFM2003-06446.

A former step is the introduction the age structure in the interaction of species with or without diffusion. The first attempts were the interaction of two species in which one of them has an age-structure and the other one has a structure independent of the age, see [8], [13], or only numeric approaches when the age-structure is considered for the two species, [15]. But as long as we know the theoretical age-dependent problem for the two species has not been tackled.

Because the interactions between the species are nonlinear, it is necessary to solve some problems for one age-dependent equation with a nonlinear reaction term. For this kind of problems, unsolved also in our knowledge, we proved the validity of the sub-supersolutions method (see [9]). So, it is the moment to profit it to approach the problem of the interaction of two species with age-structure, extending the sub-supersolutions method to systems and checking the results when the interaction of the species are the classical competition, prey-predator and symbiotic.

Our main goal in this paper is the study of the application of the sub-supersolutions method to systems age depending. The model we present perhaps it is not the most realistic possible, but it allows us to check the difficulties and advantages of the application on the cited method.

We analyze the existence and uniqueness of positive solution of the following age-dependent diffusive Lotka-Volterra systems

$$\begin{cases} u_t + u_a - \Delta u + \mu_1(x,a,t)u = u(\lambda - u + bv) & \text{in } \Omega \times \mathcal{O}, \\ v_t + v_a - \Delta v + \mu_2(x,a,t)v = v(\nu - v + cu) & \text{in } \Omega \times \mathcal{O}, \\ u(x,a,t) = v(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ u(x,a,0) = u_0(x,a), v(x,a,0) = v_0(x,a) & \text{in } \Omega \times (0, A_\dagger), \\ u(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_1(x,a,t)u(x,a,t)da & \text{in } \Omega \times (0,T), \\ v(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_2(x,a,t)v(x,a,t)da & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $A_\dagger, T > 0$, $\mathcal{O} := (0, A_\dagger) \times (0,T)$, $\lambda, \nu, b, c \in \mathbb{R}$ and

(H1) $\mu_i \in C^0(\overline{\Omega} \times [0, A_\dagger) \times [0, T]), \quad \mu_i(x, a, t) \geq 0$

$$0 < t < A_\dagger,\ x \in \Omega, \quad \lim_{a \to A_\dagger} \int_0^t \mu_i(x, a - t + \tau, \tau) d\tau = +\infty,$$
$$A_\dagger < t < T,\ x \in \Omega, \quad \lim_{a \to A_\dagger} \int_0^a \mu_i(x, \tau, t - a + \tau) d\tau = +\infty.$$

(H2) $\beta_i \in L^\infty(\Omega \times \mathcal{O}), \quad \beta_i(x, a, t) \geq 0$. We will denote $\overline{\beta}_i := \text{ess sup}_{(x,a,t) \in \Omega \times \mathcal{O}} \beta_i(x, a, t)$.

(H3) $u_0,\ v_0 \in L^2(\Omega \times (0, A_\dagger))$.

System (1.1) models the behavior of two species with densities $u(x, a, t)$ and $v(x, a, t)$ of age $a > 0$ at time $t > 0$ and at position $x \in \Omega$, which cohabit in Ω. Here μ_i and β_i denote the natural death and fertility rates of each species, respectively. The species are interacting in three different ways: if $b, c < 0$ they are competing, if $b, c > 0$ cooperating and finally, if for instance $b > 0$ and $c < 0$, u represents the predator and v the prey. In this context, b and c represent the interaction rates between the species and, finally, λ and ν are the growth rates of the species, and they are considered as parameters.

We note that hypothesis (H1) assures that the solutions u and v vanish at $a = A_\dagger$ (see [11]), and so $A_\dagger$ is the highest age attained by the individuals in the populations. In another way, the positivity of the mortality rates of species is a natural but mathematically unimportant condition as can be seen with the change of variables $w = e^{-kt}u,\ z = e^{-kt}v$ for $k > 0$ big enough.

In our knowledge, diffusive with age dependence nonlinear systems have not been analyzed deeply previously. In [1], the local existence for a system is studied when the nonlinearities satisfy some conditions out of our setting. In [16] and [17] two prey-predator systems, including the total populations in the model, are analyzed by means a fixed point theorem.

In this paper, we prove mainly the following result:

a) In the competition ($b, c < 0$), prey-predator ($bc < 0$) and weak cooperating ($b, c > 0$ and $bc < 1$) cases, there exists a unique positive solution for all $\lambda, \nu \in \mathbb{R}$ for all time $T > 0$.

b) In the strong cooperating case ($b, c > 0$ and $bc > 1$) there exists a value λ_0 such that for $\lambda, \nu > \lambda_0$ the solution blows up in finite time.

Observe that the results are in concordance with the ones obtained in the case of not age dependence, see for instance [14]. In order to prove these results, first we show that the sub-supersolution method works for this kind of systems, generalizing to systems the result obtained in the scalar case in [9]. Then, we find appropriate sub-supersolutions in each case. However, the proof of the blow-up result is more involved.

In fact, to solve the eigenvalue problem associated to (1.1), we define a compact operator whose principal eigenfunction is transformed to obtain the principal eigenfunction of our problem. We study the adjoint of the former operator, see [6], [2], [4], [3], that leads us to a backward problem whose solution has a similar transformation to build a function, φ_0^*, solution of a new backward problem also, which is used to prove the blow-up expected.

In Section 2 we build a sub-supersolution method. Section 3 is devoted to study the existence and uniqueness of positive solutions of (1.1). Finally, in Section 4 we show that in the strong cooperating case the solution blows up in finite time.

2 The sub-supersolution method

We consider the system

$$\begin{cases} u_t + u_a - \Delta u + \mu_1(x,a,t)u = f(x,a,t,u,v) & \text{in} \quad \Omega\times\mathcal{O}, \\ v_t + v_a - \Delta v + \mu_2(x,a,t)v = g(x,a,t,u,v) & \text{in} \quad \Omega\times\mathcal{O}, \\ u(x,a,t) = v(x,a,t) = 0 & \text{on} \quad \partial\Omega\times\mathcal{O}, \\ u(x,a,0) = u_0(x,a),\ v(x,a,0) = v_0(x,a) & \text{in} \quad \Omega\times(0,A_\dagger), \\ u(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_1(x,a,t)u(x,a,t)da, & \text{in} \quad \Omega\times(0,T), \\ v(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_2(x,a,t)v(x,a,t)da, & \text{in} \quad \Omega\times(0,T). \end{cases} \tag{2.1}$$

Definition 2.1 *A couple (u,v) is a solution of (2.1) if u, $v : \Omega\times\mathcal{O} \longrightarrow \mathbb{R}$ are measurable functions such that u, $v \in L^2(\mathcal{O};H_0^1(\Omega))$ and verify*

$$u_t + u_a + \mu_1 u \in L^2(\mathcal{O};H^{-1}(\Omega)), \quad v_t + v_a + \mu_2 v \in L^2(\mathcal{O};H^{-1}(\Omega))$$

$$f(\cdot,\cdot,\cdot,u,v) \in L^2(\mathcal{O};H^{-1}(\Omega)), \quad g(\cdot,\cdot,\cdot,u,v) \in L^2(\mathcal{O};H^{-1}(\Omega))$$

and for every $w \in L^2(\mathcal{O}; H_0^1(\Omega))$ *the following equalities hold*

$$\iint_{\mathcal{O}} \langle u_t + u_a + \mu_1 u, w\rangle dadt + \iiint_{\Omega\times\mathcal{O}} \nabla u \cdot \nabla w dxdadt = \iint_{\mathcal{O}} \langle f(\cdot, a, t, u, v), w\rangle dadt$$

$$\iint_{\mathcal{O}} \langle v_t + v_a + \mu_2 v, w\rangle dadt + \iiint_{\Omega\times\mathcal{O}} \nabla v \cdot \nabla w dxdadt = \iint_{\mathcal{O}} \langle g(\cdot, a, t, u, v), w\rangle dadt$$

where $\langle \cdot, \cdot \rangle$ *stands for the duality between* $H^{-1}(\Omega)$ *and* $H_0^1(\Omega)$*. Moreover, the initial conditions for the time and for the age must be verified in* $L^2(\Omega \times (0, A_\dagger))$ *and* $L^2(\Omega \times (0, T))$ *respectively.*

Definition 2.2 *Let* $u, v \in L^2(\mathcal{O}; L^2(\mathcal{O} \times \Omega))$*, with* $u \leq v$*. We define the interval* $[u, v]$ *as*

$$[u, v] := \{z \in L^2(\mathcal{O}; L^2(\mathcal{O} \times \Omega)) : \ u \leq z \leq v\}$$

We define now the suitable concept of sub-supersolutions.

Definition 2.3 *Two couples* $(\underline{u}, \overline{u})$*,* $(\underline{v}, \overline{v}) \in (L^2(\mathcal{O}, H^1(\Omega))^2$ *are a pair of sub-supersolutions of (2.1) if*

$$\underline{u} \leq \overline{u}, \qquad \underline{v} \leq \overline{v} \qquad in\ \Omega \times \mathcal{O}$$

$$f(\cdot, \cdot, \cdot, \underline{u}, v),\ f(\cdot, \cdot, \cdot, \overline{u}, v) \in L^2(\mathcal{O}, (H^1(\Omega))'),\ \forall v \in [\underline{v}, \overline{v}]$$

$$g(\cdot, \cdot, \cdot, u, \underline{v}),\ g(\cdot, \cdot, \cdot, u, \overline{v}) \in L^2(\mathcal{O}, (H^1(\Omega))'),\ \forall u \in [\underline{u}, \overline{u}]$$

Moreover, $\underline{u}$ *must verified (and analogous conditions for the other functions)*

a) $\underline{u}_t + \underline{u}_a + \mu_1 \underline{u} \in L^2(\mathcal{O}; (H^1(\Omega))')$.

b)

$$\iint_{\mathcal{O}} \langle \underline{u}_t + \underline{u}_a + \mu_1 \underline{u}, w\rangle dadt + \iiint_{\Omega\times\mathcal{O}} \nabla \underline{u} \cdot \nabla w dxdadt \leq \iint_{\mathcal{O}} \langle f(\cdot, a, t, \underline{u}, v), w\rangle dadt$$

for each $w \in L^2(\mathcal{O}, H_0^1(\Omega))$*,* $w \geq 0$ *and for each* $v \in [\underline{v}, \overline{v}]$*.*

c) $\underline{u}(x, a, t) \leq 0$ *on* $\partial\Omega \times \mathcal{O}$*.*

d) $\underline{u}(x, 0, t) \leq \int_0^{A_\dagger} \beta_1(x, a, t)\underline{u}(x, a, t)da$*, for* $(x, t) \in \Omega \times (0, T)$*.*

e) $\underline{u}(x, a, 0) \leq u_0(x, a)$*, for* $(x, a) \in \Omega \times (0, A_\dagger)$*.*

We will try to establish a sub-supersolution theorem, which will give us furthermore a priori bounds of the solutions in many cases. The result is

Theorem 2.4 *Suppose* $(H1)$, $(H2)$, $(H3)$ *and that there exists* $L > 0$ *such that*

$$\left.\begin{array}{l} |f(x,a,t,r_1,s_1) - f(x,a,t,r_2,s_2)| \leq L(|r_1 - r_2| + |s_1 - s_2|) \\ \\ |g(x,a,t,r_1,s_1) - g(x,a,t,r_2,s_2)| \leq L(|r_1 - r_2| + |s_1 - s_2|) \end{array}\right. \quad \text{a.e. } (x,a,t) \in \Omega \times \mathcal{O} \tag{2.2}$$

$\forall r_1,\ r_2 \in [u_*, u^*]$, $\quad \forall s_1,\ s_2 \in [v_*, v^*]$, *being*

$$u_* := \text{ess inf}_{\Omega\times\mathcal{O}}\underline{u}(x,a,t),\ \ u^* := \text{ess sup}_{\Omega\times\mathcal{O}}\overline{u}(x,a,t),$$

$$v_* := \text{ess inf}_{\Omega\times\mathcal{O}}\underline{v}(x,a,t),\ \ v^* := \text{ess sup}_{\Omega\times\mathcal{O}}\overline{v}(x,a,t).$$

Then, (2.1) possesses a unique solution (u, v) *such that* $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$.

Proof. We define

$$\tilde{f}(x,a,t,u,v) = f(x,a,t,\tilde{r},\tilde{s}), \quad \tilde{g}(x,a,t,u,v) = g(x,a,t,\tilde{r},\tilde{s})$$

being

$$\tilde{r} = \begin{cases} \underline{u} & \text{if} \quad u \leq \underline{u} \\ \\ u & \text{if} \quad \underline{u} \leq u \leq \overline{u} \\ \\ \overline{u} & \text{if} \quad \overline{u} \leq u \end{cases} \qquad \tilde{s} = \begin{cases} \underline{v} & \text{if} \quad v \leq \underline{v} \\ \\ v & \text{if} \quad \underline{v} \leq v \leq \overline{v} \\ \\ \overline{v} & \text{if} \quad \overline{v} \leq v \end{cases}$$

We are going to prove that problem (2.1) with $\tilde{f}$ and $\tilde{g}$ instead of f and g has a unique solution and that this solution belongs to the rectangle $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$.

If we perform the change of variables $u = e^{\alpha t}w$, $v = e^{\alpha t}z$, the problem to solve is

$$\begin{cases} w_t + w_a - \Delta w + (\mu_1(x,a,t) + \alpha)w = e^{-\alpha t}\tilde{f}(x,a,t,e^{\alpha t}w, e^{\alpha t}z) & \text{in} \quad \Omega \times \mathcal{O}, \\ \\ z_t + z_a - \Delta z + (\mu_2(x,a,t) + \alpha)z = e^{-\alpha t}\tilde{g}(x,a,t,e^{\alpha t}w, e^{\alpha t}z) & \text{in} \quad \Omega \times \mathcal{O}, \\ \\ w(x,a,t) = z(x,a,t) = 0 & \text{on} \quad \partial\Omega \times \mathcal{O}, \\ \\ w(x,a,0) = u_0(x,a),\ \ z(x,a,0) = v_0(x,a) & \text{in} \quad \Omega \times (0, A_\dagger), \\ \\ w(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_1(x,a,t)w(x,a,t)da, & \text{in} \quad \Omega \times (0,T), \\ z(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_2(x,a,t)z(x,a,t)da & \text{in} \quad \Omega \times (0,T). \end{cases} \tag{2.3}$$

We consider the space $E := L^2(\mathcal{O}, H^1(\Omega)) \times L^2(\mathcal{O}, H^1(\Omega))$ with the norm

$$\|(u,v)\|_E = (\|u\|^2_{L^2(\mathcal{O},H^1(\Omega))} + \|v\|^2_{L^2(\mathcal{O},H^1(\Omega))})^{1/2};$$

it is well known that $\|w\|^2_{L^2(\mathcal{O},H^1(\Omega))} := \iint_{\mathcal{O}} \|w(x,a,t)\|^2_{H^1(\Omega)} dadt$ and we will use (following [11]) the norm in $H^1(\Omega)$

$$\|w\|^2_\alpha := \|w\|^2_{L^2(\Omega)} + \frac{1}{\alpha}\|\nabla w\|^2_{L^2(\Omega)^N}$$

for some $\alpha > 0$. We define the map

$$\Lambda :\ E \longrightarrow E; \qquad (w,z) \longmapsto (\tilde{w}, \tilde{z})$$

being

$$\begin{cases} \tilde{w}_t + \tilde{w}_a - \Delta\tilde{w} + (L + \mu_1(x,a,t) + \alpha)\tilde{w} = e^{-\alpha t}\tilde{f}(x,a,t,e^{\alpha t}w, e^{\alpha t}z) + Lw & \text{in} \quad \Omega \times \mathcal{O}, \\ \tilde{z}_t + \tilde{z}_a - \Delta\tilde{z} + (L + \mu_2(x,a,t) + \alpha)\tilde{z} = e^{-\alpha t}\tilde{g}(x,a,t,e^{\alpha t}w, e^{\alpha t}z) + Lz & \text{in} \quad \Omega \times \mathcal{O}, \\ \tilde{w}(x,a,t) = \tilde{z}(x,a,t) = 0 & \text{on} \quad \partial\Omega \times \mathcal{O}, \\ \tilde{w}(x,a,0) = u_0(x,a),\ \ \tilde{z}(x,a,0) = v_0(x,a) & \text{in} \quad \Omega \times (0, A_\dagger), \\ \tilde{w}(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_1(x,a,t)w(x,a,t)da & \text{in} \quad \Omega \times (0,T), \\ \tilde{z}(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_2(x,a,t)z(x,a,t)da & \text{in} \quad \Omega \times (0,T). \end{cases} \tag{2.4}$$

These are two uncoupled linear problems and it is easy to see that the second members of the equations are in $L^2(\mathcal{O} \times \Omega)$. Hence, the operator is well defined.

We denote

$$Q := [e^{-\alpha t}\underline{u}, e^{-\alpha t}\overline{u}] \times [e^{-\alpha t}\underline{v}, e^{-\alpha t}\overline{v}], \quad \text{a.e.}\ (x,a,t) \in \mathcal{O} \times \Omega$$

and check that the restriction $\Lambda|_Q$ (which we will denote the same) verifies that $\Lambda :\ Q \longrightarrow Q$. In fact, let $(w,z) \in Q$ and $(\tilde{w}, \tilde{z}) := \Lambda(w,z)$; if we pose $w^* = e^{-\alpha t}\overline{u} - \tilde{w}$, it is easy to see that

$$\begin{cases} w_t^* + w_a^* - \Delta w^* + (L+\alpha+\mu_1)w^* \geq e^{-\alpha t}(f(x,a,t,\overline{u},v) - f(x,a,t,e^{\alpha t}w, e^{\alpha t}z)) + L(e^{-\alpha t}\overline{u} - w), \\ w^*(x,a,t) \geq 0, \quad \text{on } \partial\Omega \times \mathcal{O}, \\ w^*(x,a,0) \geq 0 \quad \text{in } \Omega \times (0,A_\dagger), \\ w^*(x,0,t) \geq 0 \quad \text{on } \Omega \times (0,T), \end{cases}$$

for any $v \in [\underline{v}, \overline{v}]$. We choose $v = e^{\alpha t}z$ and by the property (2.2),

$$w_t^* + w_a^* - \Delta w^* + (L+\alpha+\mu_1)w^* \geq 0.$$

From the maximum principle for these linear problems (see Lemma 2.4 in [9]) we deduce that $w^* \geq 0$, i.e, $\tilde{w} \leq e^{-\alpha t}\overline{u}$. In a analogous way the other inequalities can be proved.

We claim that with a suitable choice of α, the map is contractive. Let (w_1, z_1), $(w_2, z_2) \in E$ and

$$(w^{**}, z^{**}) := \Lambda(w_1, z_1) - \Lambda(w_2, z_2) = (\tilde{w_1} - \tilde{w_2}, \tilde{z_1} - \tilde{z_2})$$

We have to prove that

$$\exists K,\, 0 < K < 1 : \; \|(w^{**}, z^{**})\|_E \leq K\|(w_1 - w_2, z_1 - z_2)\|_E$$

But we know

$$\begin{cases} w_t^{**} + w_a^{**} - \Delta w^{**} + (L + \mu_1(x,a,t) + \alpha)w^{**} = \\ \qquad\qquad e^{-\alpha t}(\tilde{f}(x,a,t,e^{\alpha t}w_1, e^{\alpha t}z_1) - \tilde{f}(x,a,t,e^{\alpha t}w_2, e^{\alpha t}z_2)) + L(w_1 - w_2), \\ z_t^{**} + z_a^{**} - \Delta z^{**} + (L + \mu_2(x,a,t) + \alpha)z^{**} = \\ \qquad\qquad e^{-\alpha t}(\tilde{g}(x,a,t,e^{\alpha t}w_1, e^{\alpha t}z_1) - \tilde{g}(x,a,t,e^{\alpha t}w_2, e^{\alpha t}z_2)) + L(z_1 - z_2), \\ w^{**}(x,a,t) = z^{**}(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ w^{**}(x,a,0) = 0, \; z^{**}(x,a,0) = 0 & \text{in } \Omega \times (0, A_\dagger), \\ w^{**}(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_1(x,a,t)(w_1 - w_2)(x,a,t)da & \text{in } \Omega \times (0,T), \\ z^{**}(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta_2(x,a,t)(z_1 - z_2)(x,a,t)da & \text{in } \Omega \times (0,T). \end{cases} \tag{2.5}$$

Then, we take $0 < A_0 < A_\dagger$ and the test function $\phi := w^{**}\chi_{(0,A_0)}$; we multiply the first equation by ϕ, integrate on $\Omega \times \mathcal{O}_0$; $\mathcal{O}_0 := (0, A_0) \times (0, T)$, apply the integration by parts formula and it results

$$-\frac{1}{2}\overline{\beta}_1^2 A_\dagger \iiint_{\Omega\times\mathcal{O}} (w_1 - w_2)^2(x, a, t)dxdadt + \iiint_{\Omega\times\mathcal{O}_0} |\nabla w^{**}|^2 dxdadt+$$

$$\iiint_{\Omega\times\mathcal{O}_0} (L + \mu_1(x, a, t) + \alpha)|w^{**}|^2 dxdadt \le L \iiint_{\mathcal{O}_0} |w_1 - w_2|w^{**} dxdadt+$$

$$\iint_{\mathcal{O}_0} e^{-\alpha t}\langle \tilde{f}(x, a, t, e^{\alpha t}w_1, e^{\alpha t}z_1) - \tilde{f}(x, a, t, e^{\alpha t}w_2, e^{\alpha t}z_2), w^{**}\rangle.$$

But, applying (2.2) and the trivial inequality

$$(2m + n)p \le 3m^2 + 2n^2 + \frac{1}{2}p^2, \quad \forall\, m,\ n,\ p \in \mathbb{R}$$

it yields

$$L \iiint_{\mathcal{O}_0} (w_1 - w_2)w^{**} dxdadt + \iint_{\mathcal{O}_0} e^{-\alpha t}\langle \tilde{f}(x, a, t, e^{\alpha t}w_1, e^{\alpha t}z_1) - \tilde{f}(x, a, t, e^{\alpha t}w_2, e^{\alpha t}z_2), w^{**}\rangle$$

$$\le \iint_{\mathcal{O}_0} e^{-\alpha t}\|\tilde{f}(x, a, t, e^{\alpha t}w_1, e^{\alpha t}z_1) - \tilde{f}(x, a, t, e^{\alpha t}w_2, e^{\alpha t}z_2)\|_{L^2(\Omega)}\|w^{**}\|_{L^2(\Omega)} dadt+$$

$$\iint_{\mathcal{O}_0} e^{-\alpha t} L\|w_1 - w_2\|_{L^2(\Omega)}\|w^{**}\|_{L^2(\Omega)} dadt \le$$

$$\iint_{\mathcal{O}_0} (2L\|w_1 - w_2\|_{L^2(\Omega)} + L\|z_1 - z_2\|_{L^2(\Omega)})\|w^{**}\|_{L^2(\Omega)} dadt \le$$

$$\le 3L^2 \iiint_{\Omega\times\mathcal{O}_0} |w_1 - w_2|^2 dxdadt + 2L^2 \iiint_{\Omega\times\mathcal{O}_0} |z_1 - z_2|^2 dxdadt + \frac{\alpha}{2} \iint_{\mathcal{O}_0} \|w^{**}\|_\alpha^2 dadt$$

if we choose $\alpha > 1$. So,

$$-\frac{1}{2}\overline{\beta}_1^2 A_\dagger \iiint_{\Omega\times\mathcal{O}} |w_1 - w_2|^2 dxdadt + \iiint_{\Omega\times\mathcal{O}_0} |\nabla w^{**}|^2 dxdadt+$$

$$\iiint_{\Omega\times\mathcal{O}_0} (L + \mu_1(x, a, t) + \alpha)|w^{**}|^2 dxdadt \le$$

$$3L^2 \iiint_{\Omega\times\mathcal{O}_0} |w_1 - w_2|^2 dxdadt + 2L^2 \iiint_{\Omega\times\mathcal{O}_0} |z_1 - z_2|^2 dxdadt + \frac{\alpha}{2} \iint_{\mathcal{O}_0} \|w^{**}\|_\alpha^2 dadt$$

and, since $\mu_1 \ge 0$,

$$\iint_{\mathcal{O}_0} \|\nabla w^{**}\|_{L^2(\Omega)}^2 dadt + \iint_{\mathcal{O}_0} (L + \alpha)\|w^{**}\|_{L^2(\Omega)}^2 dadt - \frac{\alpha}{2} \iint_{\mathcal{O}_0} \|w^{**}\|_\alpha^2 dadt \le$$

$$(3L^2 + \frac{1}{2}\overline{\beta}_1^2 A_\dagger) \iint_{\mathcal{O}_0} \|w_1 - w_2\|_{L^2(\Omega)}^2 dadt + 2L^2 \iint_{\mathcal{O}_0} \|z_1 - z_2\|_{L^2(\Omega)}^2 dadt.$$

It is easy to see that

$$\frac{\alpha}{2}\iint_{\mathcal{O}_0}\|w^{**}\|_\alpha^2 dadt \leq (3L^2+\frac{1}{2}\overline{\beta}_1^2 A_\dagger)\|(w_1-w_2,z_1-z_2)\|_E^2$$

and taking $A_0 \to A_\dagger$,

$$\frac{\alpha}{2}\|w^{**}\|^2_{L^2(\mathcal{O},H^1(\Omega))} \leq (3L^2+\frac{1}{2}\overline{\beta}_1^2 A_\dagger)\|(w_1-w_2,z_1-z_2)\|_E^2$$

Analogously, we can obtain

$$\frac{\alpha}{2}\|z^{**}\|^2_{L^2(\mathcal{O},H^1(\Omega))} \leq (3L^2+\frac{1}{2}\overline{\beta}_2^2 A_\dagger)\|(w_1-w_2,z_1-z_2)\|_E^2$$

Therefore

$$\frac{\alpha}{2}\|(w^{**},z^{**})\|_E^2 \leq (3L^2+\frac{1}{2}\max(\overline{\beta}_1^2,\overline{\beta}_2^2)A_\dagger)\|(w_1-w_2,z_1-z_2)\|_E^2$$

and we can choose $\alpha > 1$ such that the map Λ be contractive. □

Remark 2.5 *The result can be extended for any number of equations.*

3 Applications: The Lotka-Volterra models

In this section we apply the sub-supersolution method to the systems

$$\begin{cases} u_t + u_a - \Delta u + \mu_1(x,a,t)u = u(\lambda - u + bv) & \text{in } \Omega\times\mathcal{O}, \\ v_t + v_a - \Delta v + \mu_2(x,a,t)v = v(\nu - v + cu) & \text{in } \Omega\times\mathcal{O}, \\ u(x,a,t) = v(x,a,t) = 0 & \text{on } \partial\Omega\times\mathcal{O}, \\ u(x,a,0) = u_0(x,a), v(x,a,0) = v_0(x,a) & \text{in } \Omega\times(0,A_\dagger), \\ u(x,0,t) = \displaystyle\int_0^{A_\dagger}\beta_1(x,a,t)u(x,a,t)da & \text{in } \Omega\times(0,T), \\ v(x,0,t) = \displaystyle\int_0^{A_\dagger}\beta_2(x,a,t)v(x,a,t)da & \text{in } \Omega\times(0,T), \end{cases} \tag{3.1}$$

where μ_i, β_i satisfy $(H1)$, $(H2)$, $\lambda,\nu,b,c\in\mathbb{R}$ and, instead of $(H3)$, we assume

$(H4)$ $u_0, v_0 \in L^\infty(\Omega\times(0,A_\dagger))$ and $u_0, v_0 \geq 0$.

Before studying (3.1), we need to analyze the logistic equation

$$\begin{cases} u_t + u_a - \Delta u + \mu(x,a,t)u = u(\lambda(x,a,t) - u) & \text{in } \Omega \times \mathcal{O}, \\ u(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ u(x,a,0) = u_0(x,a) & \text{in } \Omega \times (0, A_\dagger), \\ u(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta(x,a,t)u(x,a,t)da & \text{in } \Omega \times (0,T), \end{cases} \tag{3.2}$$

where μ, β and u_0 satisfy $(H1)$, $(H2)$ and $(H4)$, respectively and $\lambda \in L^\infty(\Omega \times \mathcal{O})$.

Equation (3.2) was studied in [9] under more restrictive conditions on the data. We present the main result for reader's convenience.

Proposition 3.1 *There exists a unique positive solution of (3.2), denoted by* $\Theta_{[\lambda,\mu,\beta]}$. *Moreover,* $\Theta_{[\lambda,\mu,\beta]}$ *is bounded in* $L^\infty(\Omega \times \mathcal{O})$.

Proof: We are going to find a sub-supersolution of (3.2). Take $B > 0$ such that $\beta(x,a,t) \leq B$, and consider consider the problem

$$\begin{cases} u_t + u_a - \Delta u = \overline{\lambda} u & \text{in } \Omega \times \mathcal{O}, \\ u(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\ u(x,a,0) = u_0(x,a) & \text{in } \Omega \times (0, A_\dagger), \\ u(x,0,t) = B\displaystyle\int_0^{A_\dagger} u(x,a,t)da & \text{in } \Omega \times (0,T), \end{cases} \tag{3.3}$$

where $\overline{\lambda} = \text{ess sup}_{(x,a,t)\in\Omega\times\mathcal{O}}\lambda(x,a,t)$. If we denote by $\omega_{\overline{\lambda}}$ the unique positive solution of (3.3) which is bounded, see [9], then, $(0, \omega_{\overline{\lambda}})$ is a sub-supersolution of (3.2), so that there exists a unique positive solution in $[0, \omega_{\overline{\lambda}}]$, because the lipschitzianity of $u(\lambda - u)$ in this bounded interval. But any possible solution of (3.2), u, is a subsolution of (3.3) and so $u \leq \omega_{\overline{\lambda}}$. Thus, $u = v$ and the uniqueness follows. □

Now, we are ready to state and prove the main result:

Theorem 3.2 *a) Competition case: Assume that* $b, c < 0$. *Then, there exists a unique positive solution of (3.1).*

b) Prey-predator case: Assume that $bc < 0$. *Then, there exists a unique positive solution of (3.1).*

c) *Weak cooperating case: Assume that $b, c > 0$ and $bc < 1$. Then, there exists a unique positive solution of (3.1).*

Proof: We have to build a sub-supersolution couple in each case.

Assume that $b, c < 0$. Then, it is clear that

$$(\underline{u}, \underline{v}) = (0,0), \qquad (\overline{u}, \overline{v}) = (\Theta_{[\lambda,\mu_1,\beta_1]}, \Theta_{[\nu,\mu_2,\beta_2]})$$

is a sub-supersolution of (3.1). On the other hand, if (u, v) is solution of (3.1), then u is sub-solution of (3.2) with $\mu = \mu_1$, $\beta = \beta_1$ and $\lambda \in \mathbb{R}$. Then,

$$u \leq \Theta_{[\lambda,\mu_1,\beta_1]},$$

and analogously, $v \leq \Theta_{[\nu,\mu_2,\beta_2]}$. So, any solution belongs to a bounded interval where the second members of (3.1) are Lipschitz. This concludes the uniqueness.

Assume now that $bc < 0$, for instance $b > 0$ and $c < 0$. Then, again it is not hard to show that the following couple is sub-supersolution of (3.1):

$$(\underline{u}, \underline{v}) = (0,0), \qquad (\overline{u}, \overline{v}) = (\Theta_{[\lambda+b\Theta_{[\nu,\mu_2,\beta_2]},\mu_1,\beta_1]}, \Theta_{[\nu,\mu_2,\beta_2]})$$

Observe that $\overline{u}$ is well defined because $\Theta_{[\nu,\mu_2,\beta_2]}$ is bounded. The uniqueness follows in the same way than in the competition case.

Finally, assume that $b, c > 0$ and $bc < 1$. Take one function $m \in C[0, A_\dagger]$, $B > 0$ such that

$$0 \leq m(a) \leq \mu_i(x, a, t), \qquad \beta_i \leq B, \quad i = 1, 2.$$

For this choice, let r_m the root of the equation

$$1 = B \int_0^{A_\dagger} \exp\left(ra - \int_0^a m(s)ds\right)da.$$

Denote

$$g(a) := \exp\left(-r_m a - \int_0^a m(s)ds\right).$$

and $\hat{\lambda}_1$ the principal eigenvalue of $-\Delta$ in $\hat{\Omega}$, a domain such that $\Omega \subset \hat{\Omega}$. Now, consider the pair

$$(\underline{u}, \underline{v}) = (0,0), \qquad (\overline{u}, \overline{v}) = (K_1 g(a)\hat{\varphi}_1, K_2 g(a)\hat{\varphi}_1),$$

where $K_1, K_2 > 0$ and $\hat{\varphi}_1$ is a positive eigenfunction associated to $\hat{\lambda}_1$. We are going to prove that for K_1 and K_2 large enough, the above couple is sub-supersolution of (3.1).

Indeed, it is clear that $\overline{u}, \overline{v} > 0$ on $\partial\Omega$, and $\overline{u}(a, x, 0) > u_0$ and $\overline{v}(a, x, 0) > v_0$ for large K_1 and K_2. On the other hand,

$$\begin{aligned} \overline{u}(0, x, t) = \quad & K_1\hat{\varphi}_1 = K_1\hat{\varphi}_1 \int_0^{A_\dagger} Bg(a)da \geq \\ & \int_0^{A_\dagger} \beta_1(x, a, t)K_1 g(a)\hat{\varphi}_1 da = \int_0^{A_\dagger} \beta_1(x, a, t)\overline{u}(x, a, t)da. \end{aligned}$$

Finally, it is not hard to prove that in the equations we need verify the following conditions

$$g(a)\hat{\varphi}_1(K_1 - bK_2) \geq \lambda - \hat{\lambda}_1 + r_m + m(a) - \mu_1,$$

$$g(a)\hat{\varphi}_1(K_2 - cK_1) \geq \nu - \hat{\lambda}_1 + r_m + m(a) - \mu_2.$$

Since $bc < 1$, we can choose K_1 and K_2 sufficiently large so that the above inequalities hold.

Take now a bounded solution (u, v). Then, there exist K_1 and K_2 sufficiently large such that $u \leq K_1 g(a)\hat{\varphi}_1$ and $v \leq K_2 g(a)\hat{\varphi}_1$ and $(K_1 g(a)\hat{\varphi}_1, K_2 g(a)\hat{\varphi}_1)$ supersolution of (3.1). This concludes the proof in a similar way as before. □

4 Blow-up in finite time: strong cooperating case

In the rest of the paper, our aim is to prove that in the cooperating case, when $bc > 1$ then the solution blows up in finite time. In order to prove the result, we need some previous ones and some notations.

In [10], it was proved the existence of a principal eigenvalue, denoted by λ_0, of the problem

$$\begin{cases} u_a - \Delta u + \mu(x, a)u = \lambda u & \text{in } Q := \Omega \times (0, A_\dagger), \\ u(x, a) = 0 & \text{on } \Sigma := \partial\Omega \times (0, A_\dagger), \\ u(x, 0) = \displaystyle\int_0^{A_\dagger} \beta(x, a)u(x, a)da & \text{in } \Omega, \end{cases} \tag{4.1}$$

where

$(\mathcal{H}_\mu)$ μ is a function such that $\mu \in L^\infty(\overline{\Omega} \times (0, r))$ for $r < A_\dagger$ and

$$\int_0^r \mu_M(a)da < \infty, \qquad \int_0^{A_\dagger} \mu_L(a)da = +\infty, \tag{4.2}$$

being $\mu_L(a) := \text{ess inf}_{x\in\overline{\Omega}}\mu(x, a)$ and $\mu_M(a) := \text{ess sup}_{x\in\overline{\Omega}}\mu(x, a)$.

$(\mathcal{H}_\beta)$ $\beta \in L^\infty(Q)$, $\beta \geq 0$, nontrivial and

$$\text{meas}\{a \in [0, A_\dagger] : \beta_L(a) := \text{ess inf}_{x\in\overline{\Omega}}\beta(x,a) > 0\} > 0.$$

We need recall the some points of the proof of this result. For each $\phi \in L^2(\Omega)$ we define z_ϕ the unique solution of

$$\begin{cases} z_a - \Delta z + \mu(x,a)z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(x,0) = \phi(x) & \text{in } \Omega, \end{cases} \tag{4.3}$$

and define the operator $\mathcal{B}_\lambda : L^2(\Omega) \mapsto L^2(\Omega)$ by

$$\mathcal{B}_\lambda(\phi) = \int_0^{A_\dagger} \beta(x,a)e^{\lambda a} z_\phi(x,a)da.$$

The operator $\mathcal{B}_\lambda$ is positive and compact. Denoting by $r(\mathcal{B}_\lambda)$ its spectral radius, we prove in [10] that there exists a unique value of λ, λ_0 such that $r(\mathcal{B}_{\lambda_0}) = 1$. So, by the Krein-Rutman's Theorem, there exists a positive function $\phi_0 > 0$ such that $\mathcal{B}_{\lambda_0}\phi_0 = \phi_0$. It is not difficult to prove that

$$\varphi_0 := e^{\lambda_0 a} z_{\phi_0}$$

is the eigenfunction associated to λ_0.

Again, by the Krein-Rutman's Theorem, if we denote as $\mathcal{B}^*_{\lambda_0}$ the adjoint operator of $\mathcal{B}_{\lambda_0}$, $r(\mathcal{B}^*_{\lambda_0}) = 1$, and so there exists $\psi_0^* > 0$ such that

$$\mathcal{B}^*_{\lambda_0}\psi_0^* = \psi_0^*.$$

We calculate heuristically $\mathcal{B}^*_{\lambda_0}$. For each $\psi \in L^2(\Omega)$, denote by v_ψ the unique solution (see Lemma 4.1 below) of the backward problem

$$\begin{cases} -v_a - \Delta v + \mu(x,a)v = \beta(x,a)e^{\lambda_0 a}\psi(x) & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(x, A_\dagger) = 0 & \text{in } \Omega. \end{cases} \tag{4.4}$$

Let $\phi, \psi \in L^2(\Omega)$. Then,

$$\langle \mathcal{B}_{\lambda_0}(\phi), \psi\rangle = \int_\Omega [\int_0^{A_\dagger} \beta(x,a)e^{\lambda_0 a} z_\phi(x,a)da]\psi(x)dx = \iint_Q \beta(x,a)e^{\lambda_0 a} z_\phi(x,a)\psi(x)dadx =$$

$$\iint_Q (-(v_\psi)_a - \Delta v_\psi + \mu v_\psi) z_\phi(x,a)\,da\,dx = \int_\Omega [-v_\psi(x,A_\dagger)z_\phi(x,A_\dagger) + v_\psi(x,0)z_\phi(x,0)]dx +$$

$$\iint_Q ((z_\phi)_a - \Delta z_\phi + \mu z_\phi) v_\psi\,da\,dx = \int_\Omega \phi(x) v_\psi(x,0)dx = \langle \phi, \mathcal{B}^*_{\lambda_0}(\psi)\rangle,$$

whence

$$\mathcal{B}^*_{\lambda_0}(\psi) = v_\psi(x,0).$$

Taking now

$$\varphi_0^* := e^{-\lambda_0 a} v_{\psi_0^*}$$

it is easy to prove that φ_0^* verifies

$$\begin{cases} -w_a - \Delta w + \mu(x,a)w = \lambda_0 w + \beta(x,a)w(x,0) & \text{in } Q, \\ w(x,a) = 0 & \text{on } \Sigma, \\ w(x,A_\dagger) = 0 & \text{in } \Omega. \end{cases} \tag{4.5}$$

We formalize this calculation.

Lemma 4.1 *Let $\psi \in L^2(\Omega)$ and assume*

(H5) There exists $A_0 < A_\dagger$ such that supp$(\beta) \subset \Omega \times (0, A_0)$.

There exists a unique solution $v \in L^2(Q)$ of the backward problem (4.4). Moreover, $v \in L^2(0,A_\dagger; H_0^1(\Omega))$ and there exists the value $v(x,0) \in L^2(\Omega)$.

Remark 4.2 *Hypothesis (H5) has been considered previously by many authors, [4], [5]. It has an obvious biological sense: the temporary component of the fertility rate is contained in $(0, A_0)$, i.e., in a neighborhood of $A_\dagger$ the species has not got reproductive capacity.*

Proof: Denoting by $\tilde{\mu}(x,s) := \mu(x, A_\dagger - s)$ and $\tilde{\beta}(x,s) := \beta(x, A_\dagger - s)$, system (4.4) is equivalent to

$$\begin{cases} w_s - \Delta w + \tilde{\mu}(x,s)w = \tilde{\beta}(x,s)e^{\lambda_0(A_\dagger - s)}\psi(x) & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.6}$$

and now we are interested in the value $w(x, A_\dagger)$, with the change of variable $w(x,s) = v(x, A_\dagger - s)$.

Under the change of variable

$$z = e^{-ks} w, \qquad k > 0,$$

z satisfies

$$\begin{cases} z_s - \Delta z + (\tilde{\mu} + k) z = g(x,s) := \tilde{\beta}(x,s) e^{\lambda_0 (A_\dagger - s) - ks} \psi(x) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{4.7}$$

and so by $(\mathcal{H}_\mu)$, we can take k large such that $\tilde{\mu} + k/3 \geq 0$. We study now (4.7) instead of (4.6). Define

$$\tilde{\mu}_n := \min\{\tilde{\mu}, n\}, \qquad n \in \mathbb{N},$$

and consider the problem

$$\begin{cases} z_s - \Delta z + (\tilde{\mu}_n(x,s) + k) z = g(x,s) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.8}$$

Now, for each $n \in \mathbb{N}$, since $\tilde{\mu}_n + k$ is bounded, there exists a unique z_n solution of (4.8) with

$$z_n \in C([0, A_\dagger]; H_0^1(\Omega)) \cap L^2(0, A_\dagger; H^2(\Omega) \cap H_0^1(\Omega)), \quad (z_n)_s \in L^2(0, A_\dagger; L^2(\Omega)).$$

Multiplying (4.8) by z_n and integrating we obtain

$$\frac{1}{2} \frac{d}{ds} \iint_Q |z_n|^2 + \iint_Q |\nabla z_n|^2 + \iint_Q (\tilde{\mu}_n + k) z_n^2 = \iint_Q g z_n,$$

and so, applying that $2ab \leq (\varepsilon^2 a^2 + (1/\varepsilon^2) b^2)$ we get

$$\frac{1}{2} \frac{d}{ds} \iint_Q |z_n|^2 + \iint_Q |\nabla z_n|^2 + \iint_Q (\tilde{\mu}_n + k/3) z_n^2 + (k/3) z_n^2 \leq C.$$

Now, we can extract a sequence (z_n) such that

$$\begin{aligned} & z_n \rightharpoonup z && \text{in } L^2(0, A_\dagger; H_0^1(\Omega)), \\ & \sqrt{\tilde{\mu}_n + (k/3)}\, z_n \rightharpoonup h && \text{in } L^2(Q), \\ & (z_n)_s + (\tilde{\mu}_n + k/3) z_n \rightharpoonup j && \text{in } L^2(0, A_\dagger; H^{-1}(\Omega)). \end{aligned}$$

On the other hand, for $\varphi \in C_c^\infty(0, A_\dagger; H_0^1(\Omega))$, and for n large enough, we get

$$\int_0^{A_\dagger} \langle (z_n)_s + (\tilde{\mu}_n + k/3)z_n, \varphi\rangle = \int_0^{A_\dagger} (-z_n\varphi_s + (\tilde{\mu} + k/3)z_n\varphi) \to$$

$$\to \int_0^{A_\dagger} (-z\varphi_s + (\tilde{\mu} + k/3)z\varphi) \to \int_0^{A_\dagger} (z_s + (\tilde{\mu} + k/3)z)\varphi,$$

and so

$$j = z_s + (\tilde{\mu} + k/3)z.$$

Similarly, it can be proved that $h = \sqrt{\tilde{\mu} + k/3}z$. This shows that z is solution of (4.7).

For the uniqueness, take two different solutions w_1 and w_2 of (4.6). Then, $w = w_1 - w_2$ satisfies that

$$w_s - \Delta w + \tilde{\mu}(x,s)w = 0, \qquad \text{in } Q, \quad w = 0 \quad \text{on } \Sigma, \qquad w(x,0) = 0 \quad \text{in } \Omega.$$

It suffices to multiply this problem by w and obtain that $w \equiv 0$.

Now, define

$$\tilde{\mu}_L(s) := \mu_L(A_\dagger - s), \qquad \tilde{\mu}_M(s) := \mu_M(A_\dagger - s)$$

according to (4.2). By the maximum principle, if w is solution of (4.6), then

$$w_M \le w \le w_L, \tag{4.9}$$

where w_M and w_L are the respective solutions of (4.6) with $\tilde{\mu} = \tilde{\mu}_M$ and $\tilde{\mu}_L$.

We study now (4.6) with $\tilde{\mu} = \tilde{\mu}_M$; a similar study could be made with $\tilde{\mu}_L$. If we perform the change of variable

$$z = \exp\big(\int_0^s \tilde{\mu}_M(s)ds\big)w,$$

(4.6) transforms into

$$\begin{cases} z_s - \Delta z = h(x,s) := \tilde{\beta}(x,s)\exp\big(\int_0^s \tilde{\mu}_M(\sigma)d\sigma\big)e^{\lambda_0(A_\dagger - s)}\psi(x) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{4.10}$$

But, thanks to $(H5)$

$$h(x,s) \in L^2(Q),$$

and so w_M is well-defined and $w_M \in C([0,A_\dagger];L^2(\Omega))$. This shows (4.9) and an application of convergence dominated Theorem proves that $w(x,A_\dagger) := \lim_{a\uparrow A_\dagger} w(x,a)$ is well-defined. $\square$

In order to prove the blow-up result, we need more regularity of the solution on the variable t. For that, we will use semigroup theory. Specifically, define $X := L^2(Q)$ and the operator $\mathcal{A} : X \mapsto X$ as

$$\begin{aligned}
\mathcal{A}\psi &:= -\frac{\partial \psi(x,a)}{\partial a} - \mu(x,a)\psi(x,a) + \Delta\psi(x,a), \quad \forall \psi \in D(\mathcal{A}), \text{ with} \\
D(\mathcal{A}) &:= \left\{ \psi \in X : \mathcal{A}\psi \in X, \psi\rfloor_{\partial\Omega} = 0,\ \psi(x,0) = \int_0^{A_\dagger} \beta(x,a)\psi(x,a)da \right\}.
\end{aligned}$$

In [12] it was proved, Theorem 1, that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup on the state space X.

Consider the equation

$$\begin{cases}
u_t + u_a - \Delta u + \mu(x,a)u = u(\gamma + u) & \text{in } \Omega \times \mathcal{O}, \\
u(x,a,t) = 0 & \text{on } \partial\Omega \times \mathcal{O}, \\
u(x,a,0) = u_0(x,a) & \text{in } \Omega \times (0,A_\dagger), \\
u(x,0,t) = \displaystyle\int_0^{A_\dagger} \beta(x,a)u(x,a,t)da & \text{in } \Omega \times (0,T).
\end{cases} \tag{4.11}$$

Proposition 4.3 *Assume $(\mathcal{H}_\mu)$, $(\mathcal{H}_\beta)$, $(H5)$, $\gamma > \lambda_0$ and $u_0 \in D(\mathcal{A})$. Then, there exists a unique solution of (4.11) in $(0,T)$ for some $T > 0$. Moreover, the positive solution of (4.11) blows up in finite time.*

Proof: Observe that (4.11) can be written as an evolutionary equation on X

$$\begin{cases}
u_t = \mathcal{A}u + F(u), \\
u(x,a,0) = u_0(x,a),
\end{cases} \tag{4.12}$$

where $F(u) = u(\gamma + u)$. Since, F is locally Lipschitz, it follows the existence of a local solution in $C^1([0,T];X)$, see for instance [7].

Now, let u a positive solution of (4.11), and consider

$$q(t) := \iint_Q u(x,a,t)\varphi_0^*(x,a)dadx.$$

Then, recalling that φ_0^* verifies (4.5), we get

$$q'(t) = \iint_Q u_t \varphi_0^* dadx = \iint_Q (-u_a + \Delta u - \mu u + \gamma u + u^2)\varphi_0^* dadx =$$

$$\int_\Omega [-u(x, A_\dagger)\varphi_0^*(x, A_\dagger) + u(x,0)\varphi_0^*(x,0)]dx + \gamma q + \iint_Q ((\varphi_0^*)_a + \Delta\varphi_0^* - \mu\varphi_0^*)u + u^2\varphi_0^* =$$

$$\iint_Q [\beta(x,a)\varphi_0^*(x,0) - \mu\varphi_0^* + (\varphi_0^*)_a + \Delta\varphi_0^*]u + \gamma q + \iint_Q u^2\varphi_0^* = (\gamma - \lambda_0)q + \iint_Q u^2\varphi_0^*.$$

On the other hand, by the Holder inequality

$$\iint_Q \varphi_0^* u = \iint_Q u(\varphi_0^*)^{1/2}(\varphi_0^*)^{1/2} \leq [\iint_Q u^2\varphi_0^*]^{1/2}[\iint_Q \varphi_0^*]^{1/2},$$

and so,

$$\begin{cases} q'(t) \geq (\gamma - \lambda_0)q + C_1 q^2(t), \\ q(0) = \iint_Q u_0(x,a)\varphi_0^*(x,a)dadx := q_0 > 0, \end{cases}$$

whence the blow-up follows. □

We can prove now blow-up result for the strong cooperative case.

Theorem 4.4 *Assume $b, c > 0$ and $bc > 1$. Let μ_i, β_i $i = 1, 2$ satisfying $(\mathcal{H}_\mu)$, $(\mathcal{H}_\beta)$ and (H5). Take $u_0, v_0 \in D(\mathcal{A})$ positive and $\lambda, \nu > \lambda_0$. Then, the solutions of (3.1) blow-up in finite time.*

Proof: Take $\lambda, \nu > \lambda_0$. Consider

$$K_1 := \frac{1+b}{bc-1}, \quad K_2 := \frac{1+c}{bc-1}, \quad \gamma := \min\{\lambda, \nu\} > \lambda_0, \quad \mu(x,a) := \max\{\mu_1, \mu_2\},$$

$$\beta(x,a) := \min\{\beta_1, \beta_2\} \qquad \text{and} \qquad w_0 := \min\{\frac{1}{K_1}u_0, \frac{1}{K_2}v_0\}.$$

Denote by w_γ the solution of (4.11) with the above data. Then, it is not difficult to prove that

$$(\underline{u}, \underline{v}) = (K_1 w_\gamma, K_2 w_\gamma)$$

is a sub-solution of (3.1) provided of

$$bK_2 - K_1 - 1 \geq 0 \quad \text{and} \quad cK_1 - K_2 - 1 \geq 0,$$

which is true by the definitions of K_1 and K_2. This concludes the result. □

Acknowledgements: A.S. thanks to M. Molina-Becerra for helpful comments which have improved the paper.

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